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Exact anisotropic solutions of Einstein field equations.

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Abstract: A new class of exact solutions of the system of Einstein field equations is found in a series form for an anisotropic matter. This is achieved by choosing a generalized form for one of the gravitational potentials and a particular form for the anisotropic factor. The solution of the Einstein field equations is reduced to a recurrence relation with variable rational coefficients which can be solved in general using mathematical induction. It is possible to write the new series solutions in terms of special functions. I regain well known physically reasonable model for isotropic matter.

Keywords: Einstein field equations, exact solutions, anisotropic matter

Introduction

In recent years a number of authors have studied exact solutions to the Einstein field equations corresponding to the anisotropic matter where the radial component of the pressure differs from the angular component. The gravitational filed is taken to be spherically symmetric and static since these solutions may be applied to relativistic stars. A number of researchers have examined how anisotropic matter affects critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the papers (Dev and Gleiser, 2003), (Mak and Harko, 2003), among others. Some researchers have suggested that anisotropy may be important in understanding the gravitational behavior of boson stars and the role of strange matter with densities higher than neutron stars. Mark and Harko (Mak and. Harko, 2002) and Sharma and Mukherjee (Sharma and Mukherjee, 2002) suggest that anisotropy is crucial ingredient in the description of dense stars with strange matter.

Most solutions of the Einstein field equations with anisotropic matter have been obtained in an ad hoc approach. Mainly two distinct procedures have been adopted to solve these equations for spherically symmetric static manifolds. Firstly, the coupled differential equations are solved by computations after choosing an equation of state. Secondly, the exact Einstein solutions can be obtained by specifying the geometry. I follow the later technique in an attempt to find solutions in terms of special functions that are suitable for description of relativistic stars. This approach was recently used by Chaisi and Maharaj (Chaisi. and .Maharaj, 2005) that yield a solution in terms of elementary functions. This solution have considered by many authors in the analysis of gravitational behavior of compact objects, and the study of anisotropy under strong gravitational fields. Hence the approach followed in this paper has proved to be a fruitful avenue for generating new exact solution for describing the interior spacetimes of relativistic spheres.

My intension in this paper is twofold. Firstly, I seek to model a relativistic sphere with anisotropic matter which is physically acceptable. I require that the gravitational fields and matter variables are finite, continuous and well behaved in the stellar interior and the solution is stable with respect to radial perturbations. Secondly, I seek to regain an isotropic solution of Einstein field equations which satisfy the relevant physical criteria when the anisotropy factor vanishes. This ideal is not easy to achieve in practice and only a few examples with the required two features have been found thus far. The main objective of this paper is to provide systematically a solution to Einstein equations with anisotropic matter whish satisfy the above two conditions similar to the recent treatment of Komathiraj and Maharaj (Komathiraj and. Maharaj, 2010). In Section 2, the Einstein field equations for the static spherically symmetric line element with anisotropic matter is expressed as an equivalent set of differential equations utilizing a transformation from (Durgapal and Bannerji, 1983). I chose particular forms for one of the gravitational potentials and the anisotropic factor, which enables me to obtain the condition of pressure anisotropy in the remaining gravitational potential in Section 3. This is the master equation which determines the solvability of the entire system. I assume a solution in a series form which yields recurrence relation, which I mange to solve from first principle. It is then possible to exhibit exact solutions to the Einstein field equations. I show that it is possible to express the general solutions in terms of special functions in Section 4. I regain an isotropic solution found previously by Finch and Skea (Finch and Skea, 1989).

Field equations

On physical grounds it is necessary for the gravitational field to be static and spherically symmetric. Consequently, I assume that the gravitational field of the stellar interior is represented by the line element

$$ds^2 = -e^{2f(r)}dt^2 + e^{2g(r)}dr^2 + r^2(d\theta^2 + \sin\theta\, d\phi^2) \quad [1]$$

In Schwarzschild coordinates, (t, r, θ, φ), where $f(r)$ and $g(r)$ are arbitrary functions. I consider the general case of a matter distribution with anisotropy. Therefore it is assumed that the energy momentum tensor for the interior to be an anisotropic imperfect fluid; this is represented by the form

$$T_{ij} = diag(-\mu, \; p_r, \; p_t, \; p_r) \quad [2]$$

In equation [2], the quantity μ is the energy density, P_r is the radial pressure and P_t is the tangential pressure. These physical quantities are measured relative to the commoving fluid velocity. The line element [1] and the imperfect matter distribution [2] generate the Einstein field equations; the field equations can be written in the form

$$\frac{1}{r^2}(1 - e^{-2g}) + \frac{2g'}{r} e^{-2g} = \mu \quad [3]$$

$$-\frac{1}{r^2}(1 - e^{-2g}) + \frac{2f'}{r} e^{-2g} = p_r \quad [4]$$

$$e^{-2g}\left(f'' + f'^2 + \frac{f'}{r} - f'g' - \frac{g'}{r} \right) = p_t \quad [5]$$

where primes denote differentiation with respect to r. The Einstein field equations [3] – [5] describe the gravitational behaviour for an anisotropic imperfect fluid. For matter distributions with $P_r = P_t$ (isotropic pressures), the Einstein's equations for a perfect fluid may be regained from [3] – [5]. It is convenient at this point to introduce a new independent variable x and two new functions y and Z. These are given by

$$A^2 y^2(x) = e^{2f}, \; Z(x) = e^{-2g}, \; x = Cr^2 \quad [6]$$

In equation [6], A and C are arbitrary constants. The transformation (6) was recently used by Maharaj and Komathiraj (Maharaj and Komathiraj 2007) and John and Maharaj (John and Maharaj, 2006) to describe neutron stars. The transformation [6] simplifies the field equations and the system [3] – [5] can be written as

$$\frac{1 - Z}{x} - 2\dot{Z} = \frac{\mu}{C} \quad [7]$$

$$4Z\frac{\dot{y}}{y} = \frac{Z - 1}{x} = \frac{p_r}{C} [8]$$

$$4Zx^2\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{\Delta x}{C} \right) y = 0 \quad [9]$$

$$\Delta = p_t - p_r [10]$$

where dots denote differentiations with respect to x. The quantity Δ is defined as the measure of anisotropy or anisotropy factor. The Einstein field equations as expressed in [7] – [10] is a system of four nonlinear equations in terms of six unknown variables $(\mu, P_r, P_t, Z, y, \Delta)$. The system [7] – [10] is under-determined so that there are different ways in which I can proceed with the integration process. Here I show that it is possible to specify two of the quantities and generate an ordinary differential equation in only one dependent variable in the integration process. This helps to produce a particular exact model.

Master equation

In this work, I choose physically reasonable forms for the gravitational potential Z and the measure of anisotropy Δ. Then the other gravitational potential y can be found by solving [9] which is the second order and linear in y. The remaining unknowns are then obtained from the rest of the system. I make the specific choices

$$Z = \frac{1}{1+x}, \quad \Delta = \frac{\alpha x}{1+x^2} \quad [11]$$

In equation (11), α is a real constant. The potential Z in (11) is regular at the origin and continuous in the stellar interior and the form for Δ vanishes at the centre of the star and remains continuous and bounded in the interior of the star for a wide range of values of the parameter α. Therefore the forms chosen in [11] are physically acceptable. These specific choices for Z and Δ simplify the integration process. Substitution of [11] into [9] leads to the equation

$$4(1+x)\ddot{y} - 2\dot{y} + (1-\alpha)\, y = 0 \quad [12]$$

This has the advantage of being a second order linear differential equation in the gravitational potential y. The differential equation [12] is the master equation of the system [7] – [10] and has to be solved to find exact model for an anisotropic sphere. Two categories of solutions are possible when α = 1 and α ≠ 1.

Case I: α = 1

In this case the differential equation [12] is separable and it can be immediately integrated to obtain the solution

$$y = \frac{2}{3}c_1(1+x)^{\frac{3}{2}} + c_2$$

where c_1 and c_2 are two arbitrary constants.

Case II: α ≠ 1

With α ≠ 1 the master equation [12] is difficult to solve. However it can be transformed to a different type of differential equation which can be solved using the method of Frobenius. It is convenient to introduce the new variable X = 1 + x in [12] to yield

$$4X\ddot{Y} - 2\dot{Y} + (1-\alpha)Y = 0[13]$$

where $Y = Y(X)$ is a function of the new variable X. As the point $X = 0$ is a regular singular point of [13], there exist two linearly independent solutions of the form of a power series with centre $X = 0$. These solutions can be generated using the method of Frobenius. Therefore I can assume

$$Y = \sum_{i=0}^{0} a_i X^{i+b}, a_0 \neq 0 \qquad [14]$$

In equation [14] a_i are the coefficients of the series and b is a constant. For a legitimate solution the coefficients a_i and the parameter b should be determined explicitly. On substituting [14] into [13], I obtain

$$2a_0 b(2b-3)X^{b-1} + \sum_{i=0}^{\infty}\left[(2(i+b+1)(2i+2b-1))a_{i+1} + (1-\alpha)a_i\right]X^{b+i} = 0$$

The coefficients of the various powers of X must vanish. Equating the coefficient of X^{b-1} to zero I obtain $2a_0\, b(2b - 3) = 0$. Since, $a_0 \neq 0$, $b = 0$ or $b = 3/2$. Equating the coefficient of X^{b+i} to zero, I obtain

$$a_{i+1} = \frac{1-\alpha}{[2(i+b+1)(2i+2b-1)]}a_i, \qquad i \geq 0$$

which is the recurrence formula, or difference equation, governing the structure of the solution. It is possible to express the general coefficient a_i in terms of the leading coefficient a_0 by establishing a general structure for the coefficient by considering the leading terms. These coefficients generate the pattern

$$a_{i+1} = \prod_{p=0}^{i} \frac{1-\alpha}{-2(p+b+1)(2p+2b-1)}a_0, \qquad i \geq 0$$

It is easy to establish that the result (15) holds for all positive integers p using the principle of mathematical induction.

Now it is possible to generate two linearly independent solutions to (13) with the assistance of (14) and (15). For the parameter value b=0, the first solution is given by

$$Y_1 = a_0\left[1+\sum_{i=0}^{\infty}\coprod_{p=0}^{i}\frac{1-\alpha}{-2(p+1)(2p-1)}X^{i+1}\right]$$

For the parameter value $b = 3/2$, , the second solution has the form

$$Y_2 = a_0 X^{\frac{3}{2}}\left[1+\sum_{i=0}^{\infty}\coprod_{p=0}^{i}\frac{1-\alpha}{-2(2p+5)(p+1)}X^{i+1}\right]$$

Thus the general solution to the differential equation [13], for the choice in [11], is given as

$$Y = d\ Y_1 + d_2 Y_2$$

where d_1 and d_2 are constants. In terms of the original variable x, the function Y given above becomes

$$y = A\left[1+\sum_{i=0}^{\infty}\coprod_{p=0}^{i}\frac{1-\alpha}{-2(2p+1)(2p-1)}(1+x)^{i+1}\right]+$$
$$B(1+x)^{\frac{3}{2}}\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}\frac{1-\alpha}{-2(2p+5)(p+1)}(1+x)^{i+1}\right][16]$$

where . $A = d_1a_0$, $B = d_2a_0$. Thus I have found the general series solution to differential equation [13]. Solution [16] is expressed in terms of a series with real arguments unlike the complex arguments given by software packages.

Solution in terms of special functions

The general solution [16] is given in the form of a series which define special functions. It is possible for the general solution to be written in terms of special functions in closed form which is a more desirable form for the physical description of a relativistic star. It is possible after some manipulation to write [16] in the form

$$y = A\left[1+\sum_{i=0}^{\infty}\frac{(-1)(2i+1)}{(2i+2)}[(1-\alpha)(1+x)]^{+1}\right]+$$
$$B(1+x)^{\frac{3}{2}}\left[1+\sum_{i=0}^{\infty}\frac{3(-1)^{i+1}(2i+4)}{(2i+5)}[(1-\alpha)(1+x)]^{i+1}\right][17]$$

It is interesting to observe that the equation [17] can be expressed in terms of trigonometric functions. Then it is easy to show that [17] can be written in the form

$$y = \left[A_1 + B_1\sqrt{(1-\alpha)(1+x)}\right]Sin\sqrt{(1-\alpha)(1+x)}+$$
$$\left[B_1 - A_1\sqrt{\sqrt{(1-\alpha)(1+x)}}\right]Cos\sqrt{(1-\alpha)(1+x)} \quad [18]$$

where I have set

$$A_1 = \frac{3A}{(1-\alpha)^{\frac{3}{2}}}, B_1 = B$$

Setting $\alpha = 0$ (anisotropic factor $\Delta = p_t - p_r = 0$), then [18] becomes

$$y = \left[A_1 + B_1\sqrt{1+x}Sin\sqrt{1+x}\right]+$$
$$\left[B_1 - A_1\sqrt{1+x}Cos\sqrt{1+x}\right][19]$$

The exact isotropic solution [19] is same as the model of Finch and Skea (Finch & Skea, 1989). This model satisfies all the physical conditions for an isolated spherically symmetric stellar source and consequently has been utilized by many researchers to model for neutron stars.

Discussion

I make some brief comments relating to the physical properties of the solutions found in this paper. In this work I have found a new solution [16] in the form of series and a closed form solution [18] to the Einstein field equations for an anisotropic matter distribution utilizing the method of Frobenius. The particular forms for one of the gravitational potential and the anisotropic factor were assumed. The assumed choices for the gravitational potential and the anisotropic factor in [11] are clearly positive and well behaved throughout the interior of the sphere for a wide range of the parameter values α. The general solution [16] and the special solution [17] are well defined functions on the interval [0, d] where $d=CR^2$ and R is the stellar radius. The anisotropic factor may vanish in the solution [18] and I can regain the isotropic solution [19]. Thus my approach has the advantage of necessarily containing an isotropic neutral stellar solution found previously.

Acknowledgement

I wish to express my sincere gratitude to my colleagues in the Faculty of Applied Sciences, South Eastern University of Sri Lanka for their support and attention provided during the compilation of this work.

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